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Distance-regular graphs with a_1 or c_2 at least half the valency

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ABSTRACT

In this paper, we study the distance-regular graphs Γ that have a pair of distinct vertices, say x and y, such that the number of common neighbors of x and y is about half the valency of Γ. We show that if the diameter is at least three, then such a graph, besides a finite number of exceptions, is a Taylor graph, bipartite with diameter three or a line graph.

1. Introduction

In this paper, we study the distance-regular graphs Γ that have a pair of distinct vertices, say x and y, such that the number of common neighbors of x and y is about half the valency of Γ.

To be more precise, let Γ be a distance-regular graph with valency k and diameter D. If x and y are adjacent vertices (respectively vertices at distance two), then $a_1 := a_1(x, y)$ (respectively $c_2 := c_2(x, y)$) denotes the number of common neighbors of x and y. It is known that $a_1(x, y)$ (respectively $c_2(x, y)$) does not depend on the specific pair of vertices x and y at distance one (respectively two).

Brouwer and Koolen [8] showed that if $c_2 > \frac{1}{2}k$, then $D \leqslant 3$, and $D = 3$ implies that Γ is either bipartite or a Taylor graph. In Proposition 5, we slightly extend this result.

In Theorem 16, we classify the distance-regular graphs with $a_1 \geqslant \frac{1}{2}k - 1$ and diameter at least three. Besides the distance-regular line graphs (classified by Mohar and Shawe-Taylor [14], cf. [6, Theorem 4.2.16]) and the Taylor graphs, one only finds the Johnson graph $J(7, 3)$ and the halved 7-cube. This is in some sense a generalization of the classification of the claw-free distance-regular graphs of Blokhuis and Brouwer [4], as the claw-freeness condition implies $a_1 \geqslant \frac{1}{2}k - 1$. But they

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also classified the claw-free connected non-complete strongly regular graphs (i.e., distance-regular graphs with diameter two). The classification of strongly regular graphs with $a_1 \geqslant \frac{1}{2}k - 1$ seems to be hopeless, as there are infinitely many strongly regular graphs that satisfy $a_1 \geqslant \frac{1}{2}k - 1$, but are not line graphs, for example, all the Paley graphs. A similar situation holds for the Taylor graphs. Note that the distance two graph of a non-bipartite Taylor graph is also a Taylor graph, and hence at least one of them has $a_1 \geqslant \frac{1}{2}(k-1)$ and the other one has $c_2 \geqslant \frac{1}{2}(k-1)$. If $a_1 = \frac{1}{2}(k-1)$, then it is locally a conference graph, and for any conference graph Δ, we have a Taylor graph which is locally Δ (see, for example, [6, Theorem 1.5.3]). Also there are infinitely many Taylor graphs with $a_1 \geqslant \frac{1}{2}k$, and hence there are infinitely many Taylor graphs with $c_2 \geqslant \frac{1}{2}k$ (see, for example, [6, Theorem 1.5.3] and [10, Lemma 10.12.1]).

In the last section of this paper, we will discuss the distance-regular graphs with $k_2 < 2k$, where k_2 is the number of vertices at distance two from a fixed vertex. In particular, in Theorem 17, we show that for fixed $\varepsilon > 0$, there are only finitely many distance-regular graphs with diameter at least three and $k_2 \leqslant (2-\varepsilon)k$, besides the polygons and the Taylor graphs. In Theorem 20, we classify the distance-regular graphs with $k_2 \leqslant \frac{3}{2}k$.

2. Definitions and preliminaries

All the graphs considered in this paper are finite, undirected and simple (for unexplained terminology and more details, see [6]). Suppose that Γ is a connected graph with vertex set $V(\Gamma)$ and edge set $E(\Gamma)$, where $E(\Gamma)$ consists of unordered pairs of two adjacent vertices. The *distance* $d(x, y)$ between any two vertices x and y of Γ is the length of a shortest path connecting x and y in Γ. We denote v as the number of vertices of Γ and define the *diameter* D of Γ as the maximum distance in Γ. For a vertex $x \in V(\Gamma)$, define $\Gamma_i(x)$ to be the set of vertices which are at distance precisely i from x $(0 \leqslant i \leqslant D)$. In addition, define $\Gamma_{-1}(x) = \Gamma_{D+1}(x) := \emptyset$. We write $\Gamma(x)$ instead of $\Gamma_1(x)$ and define the *local graph* $\Delta(x)$ at a vertex $x \in V(\Gamma)$ as the subgraph induced on $\Gamma(x)$. Let Δ be a graph. If the local graph $\Delta(x)$ is isomorphic to Δ for any vertex x in $V(\Gamma)$, then we say Γ is locally Δ.

A connected graph Γ with diameter D is called *distance-regular* if there are integers b_i, c_i $(0 \leqslant i \leqslant D)$ such that for any two vertices $x, y \in V(\Gamma)$ with $d(x, y) = i$, there are precisely c_i neighbors of y in $\Gamma_{i-1}(x)$ and b_i neighbors of y in $\Gamma_{i+1}(x)$, where we define $b_D = c_0 = 0$. In particular, any distance-regular graph is regular with valency $k := b_0$. Note that a (non-complete) connected *strongly regular graph* is just a distance-regular graph with diameter two. We define $a_i := k - b_i - c_i$ for notational convenience. Note that $a_i = |\Gamma(y) \cap \Gamma_i(x)|$ holds for any two vertices x, y with $d(x, y) = i$ $(0 \leqslant i \leqslant D)$.

For a distance-regular graph Γ and a vertex $x \in V(\Gamma)$, we denote $k_i := |\Gamma_i(x)|$ and $p^i_{jh} := |\{w \mid w \in \Gamma_j(x) \cap \Gamma_h(y)\}|$ for any $y \in \Gamma_i(x)$. It is easy to see that $k_i = b_0 b_1 \cdots b_{i-1}/c_1 c_2 \cdots c_i$ and hence it does not depend on x. The numbers a_i, b_{i-1} and c_i $(1 \leqslant i \leqslant D)$ are called the *intersection numbers*, and the array $\{b_0, b_1, \ldots, b_{D-1}; c_1, c_2, \ldots, c_D\}$ is called the *intersection array* of Γ. A distance-regular graph with intersection array $\{k, \mu, 1; 1, \mu, k\}$ is called a *Taylor graph*.

Suppose that Γ is a distance-regular graph with valency $k \geqslant 2$ and diameter $D \geqslant 2$, and let A_i be the matrix of Γ such that the rows and the columns of A_i are indexed by the vertices of Γ and the (x, y)-entry is 1 whenever x and y are at distance i and 0 otherwise. We will denote the adjacency matrix of Γ as A instead of A_1. The eigenvalues of the graph Γ are the eigenvalues of A.

The Bose–Mesner algebra M for a distance-regular graph Γ is the matrix algebra generated by the adjacency matrix A of Γ. A basis of M is $\{A_i \mid i = 0, \ldots, D\}$, where $A_0 = I$. The algebra M has also a basis consisting of primitive idempotents $\{E_0 = \frac{1}{v}J, E_1, \ldots, E_D\}$, and E_i is the orthogonal projection onto the eigenspace of eigenvalue θ_i. Note that M is closed under the componentwise multiplication $\circ$. The *Krein parameters* q^k_{ij} $(0 \leqslant i, j, k \leqslant D)$ of Γ are defined as

$$E_i \circ E_j = \frac{1}{v} \sum_{k=0}^{D} q^k_{ij} E_k.$$

By Delsarte's theorem (cf. [6, Theorem 2.3.2]), the Krein parameters are all non-negative.

Some standard properties of the intersection numbers are collected in the following lemma.

Lemma 1. *(See [6, Proposition 4.1.6].) Let Γ be a distance-regular graph with valency k and diameter D. Then the following holds*:

(1) $k = b_0 > b_1 \geqslant \cdots \geqslant b_{D-1}$;
(2) $1 = c_1 \leqslant c_2 \leqslant \cdots \leqslant c_D$;
(3) $b_i \geqslant c_j$ *if* $i + j \leqslant D$.

Suppose that Γ is a distance-regular graph with valency $k \geqslant 2$ and diameter $D \geqslant 1$. Then Γ has exactly $D+1$ distinct eigenvalues, namely $k = \theta_0 > \theta_1 > \cdots > \theta_D$ [6, p. 128], and the multiplicity of θ_i $(0 \leqslant i \leqslant D)$ is denote by m_i. For an eigenvalue θ of Γ, the sequence $(u_i)_{i=0,1,\ldots,D} = (u_i(\theta))_{i=0,1,\ldots,D}$ satisfying $u_0 = u_0(\theta) = 1$, $u_1 = u_1(\theta) = \theta/k$, and

$$c_i u_{i-1} + a_i u_i + b_i u_{i+1} = \theta u_i \quad (i = 1, 2, \ldots, D-1) \quad \text{and} \quad c_D u_{D-1} + a_D u_D = \theta u_D$$

is called the *standard sequence* corresponding to the eigenvalue θ [6, p. 128]. A sign change of $(u_i)_{i=0,1,\ldots,D}$ is a pair (i, j) with $0 \leqslant i < j \leqslant D$ such that $u_i u_j < 0$ and $u_t = 0$ for $i < t < j$.

In this paper we say that an intersection array is *feasible* if it satisfies the following four conditions:

(1) all its intersection numbers are integral;
(2) all the multiplicities are positive integers;
(3) for any $0 \leqslant i \leqslant D$, $k_i a_i$ is even;
(4) all Krein parameters are non-negative.

Recall that a *clique* of a graph is a set of mutually adjacent vertices and that a *co-clique* of a graph is a set of vertices with no edges. A clique $\mathcal{C}$ of a distance-regular graph with valency k, diameter $D \geqslant 2$ and smallest eigenvalue θ_D, is called *Delsarte clique* if $\mathcal{C}$ contains exactly $1 - k/\theta_D$ vertices. The *strong product* $G \boxtimes H$ of graphs G and H is a graph such that the vertex set of $G \boxtimes H$ is the Cartesian product $V(G) \times V(H)$ and any two different vertices (u, v) and (u', v') are adjacent in $G \boxtimes H$ if and only if ($u = u'$ or u is adjacent to u') and ($v = v'$ or v is adjacent to v'). For a given positive integer s, the *s-clique extension* of a graph G is the strong product $G \boxtimes K_s$ of G and K_s, where K_s is the complete graph (or clique) of size s.

A graph Γ is called *graph of order* (s, t) if Γ is locally disjoint union of $t+1$ copies of K_s. Note that, if Γ is a distance-regular graph with $c_2 = 1$ and valency k, then Γ is a graph of order (s, t) for some $s (= a_1 + 1)$ and t, and hence the valency k is equal to $s(t+1)$. A *Terwilliger graph* is a connected non-complete graph Γ such that, for any two vertices u, v at distance two, the subgraph induced on $\Gamma(u) \cap \Gamma(v)$ in Γ is a clique of size μ (for some fixed $\mu \geqslant 1$). A graph Γ is called *bipartite* if it has no odd cycle. (If Γ is a distance-regular graph with diameter D and bipartite, then $a_1 = a_2 = \cdots = a_D = 0$.)

An *antipodal* graph is a connected graph Γ with diameter $D > 1$ for which being at distance 0 or D is an equivalence relation between vertices. If, moreover, all equivalence classes have the same size r, then Γ is also called an *antipodal r-cover.*

Recall the following interlacing result.

Theorem 2. *(Cf. [11].) Let $m \leqslant n$ be two positive integers. Let A be an $n \times n$ matrix, that is similar to a (real) symmetric matrix, and let B be a principal $m \times m$ submatrix of A. Then, for $i = 1, \ldots, m$,*

$$\theta_{n-m+i}(A) \leqslant \theta_i(B) \leqslant \theta_i(A)$$

holds, where A has eigenvalues $\theta_1(A) \geqslant \theta_2(A) \geqslant \cdots \geqslant \theta_n(A)$ and B has eigenvalues $\theta_1(B) \geqslant \theta_2(B) \geqslant \cdots \geqslant \theta_m(B)$.

For the convenience of the reader, we give a proof of the following lemma.

Lemma 3. *(See [8, Lemma 3.1].) Let Γ be a distance-regular graph with diameter D and valency k. If $D \geqslant 3$, then $b_1 \geqslant \frac{1}{3}k + \frac{1}{3}$.*

Proof. If $b_1 < \frac{1}{3}k + \frac{1}{3}$, then $a_1 + 1 > \frac{2}{3}k - \frac{1}{3}$. Let x be a vertex of Γ. As $\Delta(x)$ is not a complete graph, $\Delta(x)$ has non-adjacent vertices. So, $2(a_1+1) - (c_2 - 1) \leqslant k$, and hence $c_2 \geqslant 2(a_1+1) - k + 1 > \frac{1}{3}k + \frac{1}{3} > b_1$. Lemma 1(3) implies that $D \leqslant 2$ and this is a contradiction. □

In [6, Proposition 5.5.1(ii)], there is an error in the statement when equality occurs. Although this is fixed in [5], for the convenience of the reader, we give a proof for this.

Proposition 4. *Let Γ be a distance-regular graph with diameter $D \geqslant 3$, valency k and intersection number $a_1 > 0$. Then $a_i + a_{i+1} \geqslant a_1$ for $i = 1, \dots, D-1$. If $a_i + a_{i+1} = a_1$, then $i = D-1$, $a_D = 0$, $a_{D-1} = a_1$ and $b_{D-1} = 1$.*

Proof. Let $i \in \{1, \dots, D-1\}$ and let x, y be a pair of vertices of Γ at distance $i+1$. Then $|\Gamma_i(x) \cap \Gamma_2(y)| = p^{i+1}_{i,2} = c_{i+1}(a_i + a_{i+1} - a_1)/c_2 \geqslant 0$ which gives $a_i + a_{i+1} \geqslant a_1$. If $a_i + a_{i+1} = a_1$, then we have $i = D-1$ by [6, Proposition 5.5.1(i)], and hence $p^D_{D-1,2} = 0$. This implies that each vertex of $\Gamma(y) \cap \Gamma_{D-1}(x)$ is adjacent to each vertex of $\Gamma(y) \cap \Gamma_D(x)$. We will show that $a_D = 0$ by way of contradiction. Assume $a_D > 0$. This implies that the complement of the local graph at y, say $\overline{\Delta(y)}$, is disconnected. Let C be a connected component of $\overline{\Delta(y)}$. If C is a singleton, then $a_1 = k-1$, and hence Γ is complete. So we have $|C| \geqslant 2$. Let z and w be two adjacent vertices in C. Then they have at most $c_2 - 1$ common neighbors in $\Delta(y)$, as z and w are not adjacent in Γ. This means that C has size at least $k - c_2 + 1$. This means, $k = |\overline{\Delta(y)}| \geqslant 2(k - c_2 + 1)$, and hence $c_2 \geqslant \frac{1}{2}k + 1$. On the other hand, $\overline{\Delta(y)}$ is $(k - a_1 - 1)$-regular, and hence C has size at least $k - a_1$ and we obtain $k = |\overline{\Delta(y)}| \geqslant 2(k - a_1)$. This means that $a_1 \geqslant \frac{1}{2}k$, and hence $b_1 \leqslant \frac{1}{2}k - 1 < c_2$. This contradicts $D \geqslant 3$. Therefore $a_D = 0$. Now we show that $b_{D-1} = 1$. Let z be a vertex of $\Gamma_{D-1}(x)$. If $b_{D-1} > 1$, then for any two vertices u and v of $\Gamma(z) \cap \Gamma_D(x)$, we have $a_2 = |\Gamma(u) \cap \Gamma_2(v)| = 0$ as $a_D = 0$ and $p^D_{D-1,2} = 0$. But, by [6, Proposition 5.5.6], we have $a_2 \geqslant \min\{b_2, c_2\} \geqslant 1$, as $a_1 > 0$ and $D \geqslant 3$. This is a contradiction. So, $b_{D-1} = 1$. □

3. Distance-regular graphs with large a_1

In this section we classify the distance-regular graphs with $a_1 \geqslant \frac{1}{2}k - 1$ and diameter $D \geqslant 3$. First we show that if c_2 or b_2 is large, then $D = 3$ and imprimitive, where imprimitive means that the graphs are either bipartite or antipodal. This generalizes [8, Lemma 3.14].

Proposition 5. *Let Γ be a distance-regular graph with diameter $D \geqslant 3$ and valency k. If $c_2 > \frac{1}{2}k$ or $b_2 > \frac{1}{2}k_3$, then $D = 3$ and Γ is either bipartite or a Taylor graph.*

Proof. Assume $c_2 > \frac{1}{2}k$ or $b_2 > \frac{1}{2}k_3$. Then $d(y, w) \leqslant 2$ for a fixed vertex x and $y, w \in \Gamma_2(x)$. Then $p^2_{23} = 0$ which in turn implies $c_3(a_2 + a_3 - a_1)/c_2 = p^3_{22} = 0$ and hence $a_1 = a_2 + a_3$. If $a_1 \neq 0$, then, by Proposition 4, $D = 3$ and Γ is a Taylor graph. So, we may assume $a_1 = 0$. Then $a_2 = a_3 = 0$. Furthermore, if $b_2 > \frac{1}{2}k_3$ and $c_2 \leqslant \frac{1}{2}k$, then $k_3 = kb_1b_2/c_2c_3 > k(k-1)(k_3/2)/(k/2)c_3 = (k-1)k_3/c_3$. So, $c_3 > k-1$ and hence $D = 3$ and Γ is bipartite. This shows that we may assume $c_2 > \frac{1}{2}k$ but this again implies $D = 3$ (and Γ is bipartite) by Lemma 1(3) as $c_2 > b_2$. □

Now we show that the distance-regular Terwilliger graphs with large a_1 and $c_2 \geqslant 2$ are known.

Proposition 6.

(1) *Let Γ be a connected non-complete strongly regular graph with valency k. If $c_2 = 1$ and $k < 7(a_1 + 1)$, then Γ is either the pentagon or the Petersen graph.*
(2) *Let Γ be a connected non-complete strongly regular Terwilliger graph with v vertices and valency k. If $v \leqslant 7k$ then Γ is either the pentagon or the Petersen graph.*

(3) *Let Γ be a distance-regular Terwilliger graph with v vertices, valency k and diameter D. If $k \leqslant (6 + \frac{8}{57})(a_1+1)$ and $c_2 \geqslant 2$, then Γ is the icosahedron, the Doro graph (see* [6, Section 12.1]) *or the Conway–Smith graph (see* [6, Section 13.2]).

Proof. (1): Since $c_2 = 1$, then $a_1 + 1$ divides k, and we obtain $(t+1)(a_1+1) = k < 7(a_1+1)$ with $t \in \{1,2,3,4,5\}$. Now we will show that $a_1 < t$. Let $\mathbf{C}$ be the set of all (a_1+2)-cliques in the graph Γ. By counting the number of pairs $(x, \mathcal{C})$, where x is a vertex of the graph Γ and $\mathcal{C}$ is a clique of $\mathbf{C}$ containing x, in two ways, we have $|V(\Gamma)|(t+1) = (a_1+2)|\mathbf{C}|$.

Suppose that $a_1 \geqslant t$, then $|\mathbf{C}| < |V(\Gamma)|$. Let M be the vertex-((a_1+2)-clique) incidence matrix of Γ, i.e., M is the 01-matrix whose rows and columns are indexed by the vertices and (a_1+2)-cliques of Γ, respectively, and the $(x,\mathcal{C})$-entry of M is 1 whenever the vertex x is in the clique $\mathcal{C}$ and 0 otherwise. Then MM^T is a singular matrix, as $|\mathbf{C}| < |V(\Gamma)|$, and hence $-t-1$ is an eigenvalue of Γ, as $MM^T = A + (t+1)I$, where A is the adjacency matrix of Γ. As $-t-1$ is an eigenvalue of Γ, by [6, Proposition 4.4.6], any clique $\mathcal{C}$ of the set $\mathbf{C}$ is a Delsarte clique, and hence for all $x \in V(\Gamma)$ and all $\mathcal{C} \in \mathbf{C}$, there exist $y \in \mathcal{C}$ such that $d(x,y) \leqslant 1$ by [1, Lemma 3]. By considering two vertices u and v at distance two and $t+1$ cliques containing u, it follows that u and v have at least $t+1$ common neighbors, which is a contradiction.

So, $0 \leqslant a_1 < t \leqslant 5$ holds. But except for the cases $(t,a_1) = (1,0)$ and $(t,a_1) = (2,0)$, a strongly regular graph does not exist, as the multiplicity of the second largest eigenvalue is not an integer. For $(t,a_1) = (1,0)$ and $(t,a_1) = (2,0)$, we obtain the pentagon and the Petersen graph, respectively.

(2): As $1 + k + b_1k/c_2 < v \leqslant 7k$, it follows that $b_1 < 6c_2$. If $c_2 = 1$, then by (1), Γ is either the pentagon or the Petersen graph. So, we may assume that $c_2 > 1$. Note that $k \geqslant 2(a_1+1) - (c_2-1)$ (as Γ is not a complete graph), which implies that $13c_2 > c_2 + 2b_1 \geqslant k+1$, and hence $k < 13c_2 - 1 < 50(c_2-1)$, as $c_2 \geqslant 2$. So, there are no such strongly regular Terwilliger graph with $c_2 > 1$ by [6, Corollary 1.16.6].

(3): Let x be a vertex of Γ. Then the local graph $\Delta(x)$ at x is an s-clique extension of a strongly regular Terwilliger graph Σ with parameters $\bar{v} = k/s$, $\bar{k} = (a_1 - s + 1)/s$, and $\bar{c_2} = (c_2-1)/s$ by [6, Theorem 1.16.3]. As $\bar{c_2} \geqslant 1$ ($c_2 \geqslant 2$), we have $c_2 - 1 \geqslant s$. If Σ is the pentagon or the Petersen graph, then $\Delta(x) = \Sigma$ and $s = 1$, and hence by [6, Theorem 1.16.5], we are done in this case. So we may assume that $\bar{v} > 7\bar{k}$ (by (2)). As $k \leqslant (6+\frac{8}{57})(a_1+1)$, we obtain $s > \frac{7}{57}(a_1+1) \geqslant \frac{1}{50}k$. Now, as $c_2 - 1 \geqslant s$, it follows $k < 50(c_2-1)$, and hence we are done by [6, Corollary 1.16.6]. □

Remark 7. There exist generalized quadrangles of order (q,q) for any prime power q (see, for example, [10, p. 83]). Note that the flag graph of any generalized quadrangle of order (q,q) is a distance-regular graph with $k = 2q$ and $c_2 = 1$. This shows that there are infinitely many distance-regular graphs with $a_1 > \frac{1}{7}k$. See also, Theorem 16 below.

Before we classify the distance-regular graphs with $a_1 \geqslant \frac{1}{2}k - 1$, we first introduce some results for this classification.

Lemma 8. *Let Γ be a distance-regular graph with diameter $D \geqslant 3$ and valency k. If $a_1 \geqslant \frac{1}{2}k - 1$ and $c_2 \geqslant 2$, then $b_2 < c_2$, and hence $D = 3$.*

Proof. If Γ is a Terwilliger graph, then Γ is the icosahedron by Proposition 6(3). If Γ is not a Terwilliger graph, then Γ has a quadrangle. Then by [6, Theorem 5.2.1], $c_2 - b_2 \geqslant c_1 - b_1 + a_1 + 2 \geqslant 2$ holds, and hence $D = 3$ by Lemma 1(3). □

3.1. Some results on eigenvalues

In the next three lemmas, we give some results on the eigenvalues of a distance-regular graph.

Lemma 9. *Let Γ be a distance-regular graph with diameter three and distinct eigenvalues $k = \theta_0 > \theta_1 > \theta_2 > \theta_3$. If $a_3 = 0$, then $\theta_1 > 0 > -1 \geqslant \theta_2 \geqslant -b_2 \geqslant \theta_3$.*

Proof. As $a_3 = 0$, we know that θ_1, θ_2 and θ_3 are the eigenvalues of

$$T := \begin{bmatrix} -1 & b_1 & 0 \\ 1 & k - b_1 - c_2 & b_2 \\ 0 & c_2 & -b_2 \end{bmatrix}$$

by [6, p. 130]. Since the principal submatrix $\left[\begin{smallmatrix} -1 & 0 \\ 0 & -b_2 \end{smallmatrix}\right]$ of T has eigenvalues -1 and $-b_2$, it follows that the inequality $\theta_1 \geqslant -1 \geqslant \theta_2 \geqslant -b_2 \geqslant \theta_3$ holds by Theorem 2. As Γ has an induced path P of length three, $\theta_1 \geqslant$ second largest eigenvalue of P, which is greater than zero. $\square$

Lemma 10. *Let Γ be a distance-regular graph with diameter $D \geqslant 3$ and distinct eigenvalues $k = \theta_0 > \theta_1 > \cdots > \theta_D$. If Γ has an eigenvalue θ with multiplicity smaller than $\frac{1}{2}k$ then the following holds*:

(1) $\theta \in \{\theta_1, \theta_D\}$,
(2) θ *is integral,*
(3) $\theta + 1$ *divides* b_1.

Proof. This lemma follows immediately from [6, Theorem 4.4.4]. $\square$

Lemma 11. *Let Γ be a distance-regular graph with diameter $D \geqslant 3$ and distinct eigenvalues $k = \theta_0 > \theta_1 > \cdots > \theta_D$. If Γ has an eigenvalue θ with multiplicity at most $k - 2$ then the following holds*:

(1) $\theta_2 \geqslant -1 - b_1/(\theta + 1)$ *if* $\theta = \theta_1$,
(2) $\theta_{D-1} \leqslant -1 - b_1/(\theta + 1)$ *if* $\theta = \theta_D$.

Proof. Let x be a vertex of Γ. Then the local graph $\Delta(x)$ has smallest eigenvalue at least $-1 - b_1/(\theta_1 + 1)$ and second largest eigenvalue at most $-1 - b_1/(\theta_D + 1)$, by [6, Theorem 4.4.3]. As $\Delta(x)$ has k eigenvalues at least $-1 - b_1/(\theta_1 + 1)$ and $k - 1$ eigenvalues at most $-1 - b_1/(\theta_D + 1)$, by Theorem 2, the inequalities follow. $\square$

Lemma 11 gives evidence for the following conjecture.

Conjecture 12. *Let Γ be a distance-regular graph with diameter three and distinct eigenvalues $k = \theta_0 > \theta_1 > \theta_2 > \theta_3$. Then*

$$-1 - \frac{b_1}{\theta_3 + 1} \geqslant \theta_2 \geqslant -1 - \frac{b_1}{\theta_1 + 1}.$$

Remark 13. Conjecture 12 is true when Γ is bipartite, as then $b_1 = k - 1$, $\theta_1 = -\theta_2 = \sqrt{b_2}$ and $\theta_3 = -k$. Hence $0 = -1 - b_1/(\theta_3 + 1) > \theta_2 = -\sqrt{b_2} > -1 - b_1/(\theta_1 + 1)$, where the last inequality holds, as $k > b_2$. Conjecture 12 is also true when Γ is antipodal, as then $\theta_2 = -1$. We also checked that all the feasible intersection arrays in the table of primitive distance-regular graphs with diameter three [6, pp. 425–431], satisfy this conjecture.

Remark 14. For a distance-regular graph with diameter $D \geqslant 4$, we have $\theta_2 \geqslant 0$ by Theorem 2.

Remark 15. Fiol and Garriga [9] obtained some properties of the spectrum of a connected graph with four distinct eigenvalues diameter three, under which the distance-regular graphs with diameter three fall.

3.2. Classification of distance-regular graphs with $a_1 \geqslant \frac{1}{2}k - 1$

Now we are ready to classify the distance-regular graphs with $a_1 \geqslant \frac{1}{2}k - 1$.

Theorem 16. *Let Γ be a distance-regular graph with diameter $D \geqslant 3$ and valency k. If $a_1 \geqslant \frac{1}{2}k - 1$, then one of the following holds*:

(1) *Γ is a polygon,*
(2) *Γ is the line graph of a Moore graph,*
(3) *Γ is the flag graph of a regular generalized D-gon of order (s, s) for some s,*
(4) *Γ is a Taylor graph,*
(5) *Γ is the Johnson graph $J(7, 3)$,*
(6) *Γ is the halved 7-cube.*

Proof. If Γ does not contain a $K_{1,3}$ as an induced subgraph of Γ, then one of (1), (2), (3) or (4) holds by [4, Theorem 1.2]. So, we may assume that Γ contains a $K_{1,3}$. Since Γ contains a $K_{1,3}$, we find $3(a_1 + 1) - 3(c_2 - 1) \leqslant k$, and hence $c_2 \geqslant \frac{1}{6}k + 1$. So, $c_2 \geqslant 2$ and it follows that $D = 3$ by Lemma 8. We will consider two cases, namely the case $a_3 = 0$ and the case $a_3 \neq 0$.

Case 1): First let us assume $a_3 = 0$. Then $c_3 = k$ and it follows $k_3 = kb_1b_2/c_2c_3 = b_1b_2/c_2$. Let x be a vertex of Γ and y be a vertex of $\Gamma_3(x)$. Then $\Gamma_3(x)$ contains $k(b_2 - 1)/c_2$ vertices which are at distance two from y, as $a_3 = 0$. Hence $b_1b_2 > k(b_2 - 1)$ and this implies $b_2 = 1$, as $b_1 \leqslant \frac{1}{2}k$. Thus, $k_3 \in \{1, 2\}$, as $b_1 \leqslant \frac{1}{2}k$ and $c_2 > \frac{1}{6}k$.

(1) If $k_3 = 1$, then Γ is a Taylor graph.
(2) If $k_3 = 2$, then $b_1 = 2c_2$, and hence $v = 3k + 3$, as $k_2 = 2k$. Since $b_1 \leqslant \frac{1}{2}k$, we have $c_2 \leqslant \frac{1}{4}k$, which implies $a_2 \geqslant \frac{3}{4}k - 1$, as $b_2 = 1$. Let $k > \theta_1 > \theta_2 > \theta_3$ be the distinct eigenvalues of Γ. Then $k + \theta_1 + \theta_2 + \theta_3 = a_1 + a_2 + a_3 \geqslant (\frac{1}{2}k - 1) + (\frac{3}{4}k - 1)$ implies $\theta_1 + \theta_2 + \theta_3 \geqslant \frac{1}{4}k - 2$. As $a_3 = 0$ and $b_2 = 1$, we have $\theta_2 = -1 > \theta_3$ by Lemma 9, and hence $\theta_1 > \frac{1}{4}k$.
If the multiplicity m_1 of θ_1 is smaller than $\frac{1}{2}k$, then by Lemma 10, $b_1/(\theta_1 + 1)$ is an integer and hence $\theta_1 = b_1 - 1$, as $b_1 \leqslant \frac{1}{2}k$ and $\theta_1 > \frac{1}{4}k$. Then there is no such a graph by [6, Theorem 4.4.11]. If $m_1 \geqslant \frac{1}{2}k$, then $4k^2 \geqslant (3k + 3)k = vk = k^2 + m_1\theta_1^2 + m_2\theta_2^2 + m_3\theta_3^2 > k^2 + \frac{1}{2}k(\frac{1}{4}k)^2$, and hence $k < 96$.
We checked by computer the feasible intersection arrays of antipodal 3-covers with diameter three, satisfying $k < 96$, $a_1 \geqslant \frac{1}{2}k - 1$ and $c_2 > \frac{1}{6}k$, and no intersection arrays were feasible .

Case 2): Now we assume $a_3 \neq 0$. We first will show that in this case $k \leqslant 945$ holds. If $\theta_1 \geqslant \frac{1}{4}k$ holds, then similarly as in (2) of *Case* 1), we can show that $k < 96$ or $\theta_1 = b_1 - 1$. If $\theta_1 = b_1 - 1 \geqslant \frac{1}{4}k$ holds, then Γ is either the Johnson graph $J(7, 3)$ or the halved 7-cube by [6, Theorem 4.4.11]. So, we find that if $\theta_1 \geqslant \frac{1}{4}k$ holds, then we have $k < 96$. So we may assume that $\theta_1 < \frac{1}{4}k$. As $\theta_1 \geqslant \min\{\frac{1}{2}(a_1 + \sqrt{a_1^2 + 4k}), a_3\}$ [12, Lemma 6] and $\frac{1}{2}(a_1 + \sqrt{a_1^2 + 4k}) > \frac{1}{4}k$, we have $a_3 < \frac{1}{4}k$, and hence $c_3 > \frac{3}{4}k$. As $a_3 \neq 0$, Γ is not a Taylor graph. Then by Proposition 5, we find $\frac{1}{2}(k/c_2) \cdot k \geqslant (b_1/c_2) \cdot k = k_2 \geqslant 2c_3 > \frac{3}{2}k$, which implies $c_2 < \frac{1}{3}k$. Let $\eta_1 = \max\{-1, a_3 - b_2\}$ and $\eta_2 = \min\{-1, a_3 - b_2\}$. Then $\theta_1 \geqslant \eta_1 \geqslant \theta_2 \geqslant \eta_2 \geqslant \theta_3$ by [13, Proposition 3.2]. Now we will show that $\theta_1 \geqslant \frac{k}{2} - c_2$.

If $\theta_2 \geqslant 0$, then $\eta_1 = a_3 - b_2$ and $\eta_2 = -1$, and hence $a_3 \geqslant b_2 + \theta_2$ and $\theta_3 \leqslant -1$. We find $\theta_1 + \theta_2 + \theta_3 = a_1 + a_2 + a_3 - k \geqslant -\frac{1}{2}k - 1 + a_2 + b_2 + \theta_2$ and this implies $\theta_1 \geqslant a_2 + b_2 - \frac{1}{2}k = \frac{1}{2}k - c_2$.

If $\theta_2 < 0$, then $\theta_3 \leqslant a_3 - b_2$ implies $a_3 \geqslant b_2 + \theta_3$. We find $\theta_1 + \theta_2 + \theta_3 = a_1 + a_2 + a_3 - k \geqslant -\frac{1}{2}k - 1 + a_2 + b_2 + \theta_3$ and this implies $\theta_1 \geqslant a_2 + b_2 - \frac{1}{2}k = \frac{1}{2}k - c_2$ as $\theta_2 \leqslant -1$ or $a_3 - b_2 \geqslant \theta_2 \geqslant -1 \geqslant \theta_3$. So, we have shown $\theta_1 \geqslant \frac{1}{2}k - c_2$ and this implies $\theta_1 > \frac{1}{6}k$ and $c_2 > \frac{1}{4}k$, as $c_2 < \frac{1}{3}k$ and $\theta_1 < \frac{1}{4}k$ respectively.

If the multiplicity m_1 of θ_1 is smaller than $\frac{1}{2}k$, then by Lemma 10, $b_1/(\theta_1 + 1)$ is an integer and hence $b_1/(\theta_1 + 1) \in \{1, 2\}$, as $b_1 \leqslant \frac{1}{2}k$ and $\theta_1 > \frac{1}{6}k$.

If $b_1/(\theta_1 + 1) = 1$, then there is no such a graph by [6, Theorem 4.4.11].

If $b_1/(\theta_1 + 1) = 2$, then $u_2(\theta_1) = ((\theta_1 - a_1)u_1(\theta_1) - 1)/b_1 = (\frac{3}{4}b_1 - \frac{1}{2}k - \frac{3}{2})/k < -\frac{1}{8}$, as $\theta_1 - a_1 = \frac{1}{2}b_1 - 1 - a_1 = \frac{3}{2}b_1 - k$ and $u_1(\theta_1) = \theta_1/k = (\frac{1}{2}b_1 - 1)/k$. As $\theta_1 = a_3$ implies $u_2(\theta_1) = 0$, we assume $\theta_1 \neq a_3$. Then $c_3u_2(\theta_1) + a_3u_3(\theta_1) = \theta_1u_3(\theta_1)$ follows that $u_3(\theta_1) = (c_3/(\theta_1 - a_3)) \cdot u_2(\theta_1)$. As $u_3(\theta_1) \geqslant -1$, we find $\theta_1 - a_3 > \frac{3}{32}k$ and hence $a_3 < \frac{5}{32}k$. As $\frac{1}{2}b_1 - 1 = \theta_1 \geqslant \frac{1}{2}k - c_2$ and $k_2 \geqslant 2c_3$, we find that $\theta_1 \leqslant \frac{9}{40}k$ implies $c_3 \leqslant \frac{37}{44}k$, unless $k \leqslant 160$. This shows that $k \leqslant 160$ or $\theta_1 > \frac{9}{40}k$. So, we may assume $\theta_1 > \frac{9}{40}k$ and $a_3 < \frac{5}{32}k$. As $a_1 \geqslant \frac{1}{2}k - 1$ and $c_2 \geqslant 2$, we have $b_2 < c_2$ and this shows $k_3 = kb_1b_2/c_2c_3 < kb_1/c_3 < k(k/2)/(27k/32) = \frac{16}{27}k$. Then by [6, Theorem 4.1.4], we find

$$m_1 = \frac{v}{\sum_{i=0}^{3} u_i(\theta_1)^2 k_i} < \frac{k + k_2 + 16k/27}{(9/40)^2k + (-1/8)^2k_2} = \frac{43k/27 + k_2}{81k/1600 + k_2/64}.$$

As $\frac{1}{3}k \leqslant b_1 \leqslant \frac{1}{2}k$ and $\frac{1}{4}k < c_2 < \frac{1}{3}k$, we have $k \leqslant k_2 < 2k$ and $(43k/27 + k_2)/(81k/1600 + k_2/64)$ has maximum value smaller than 43.88 when $k_2 = 2k$. So, $m_1 \leqslant 43$. Thus, we have $k \leqslant \frac{1}{2}(43+2)(43-1) = 945$ by [6, Theorem 5.3.2].

If $m_1 \geqslant \frac{1}{2}k$, then similarly as in (2) of *Case* 1) we obtain $k < 312$.

We checked by computer the feasible intersection arrays of distance-regular graphs with diameter three satisfying $k \leqslant 945$, $a_1 \geqslant \frac{1}{2}k - 1$, $c_2 > \frac{1}{6}k$ and $a_3 \neq 0$, and only the intersection arrays of the Johnson graph $J(7,3)$ and the halved 7-cube, were feasible. □

4. Distance-regular graphs with small k_2

In this section we give two results on the distance-regular graphs with small k_2. In the first result we look at $k_2 < 2k$, and in the second result we classify the distance-regular graphs with $k_2 \leqslant \frac{3}{2}k$ and diameter at least three.

Theorem 17. *Let $\varepsilon > 0$. Then there exist a real number $\kappa = \kappa(\varepsilon) \geqslant 3$ such that if Γ is a distance-regular graph with diameter $D \geqslant 3$, valency $k \geqslant \kappa(\varepsilon)$ and $k_2 \leqslant (2-\varepsilon)k$, then $D = 3$ and Γ is either bipartite or a Taylor graph.*

Proof. We may assume $k \geqslant 3$. If $c_2 = 1$, then $b_1 = 1$, as $k \leqslant k_2 = kb_1/c_2 \leqslant (2-\varepsilon)k$ and this implies $a_1 = k-2$. As $c_2 = 1$, we know that $a_1 + 1 = k-1$ divides k, which gives $k = 2$, which contradicts $k \geqslant 3$. So, we may assume $c_2 \geqslant 2$. Suppose Γ either has $D \geqslant 4$ or ($D = 3$ and Γ is not bipartite or a Taylor graph), then by Proposition 5, $c_2 \leqslant \frac{1}{2}k$ and $b_2 \leqslant \frac{1}{2}k_3$. As $c_2 \leqslant \frac{1}{2}k$ (respectively $b_2 \leqslant \frac{1}{2}k_3$), we have $(2-\varepsilon)k \geqslant k_2 \geqslant 2b_1$ (respectively $(2-\varepsilon)k \geqslant k_2 \geqslant 2c_3$), and hence $a_1 \geqslant \frac{1}{2}\varepsilon k - 1$ (respectively $a_3 \geqslant \frac{1}{2}\varepsilon k$). So, by [12, Lemma 6], we have $\theta_1 \geqslant \min\{\frac{1}{2}(a_1 + \sqrt{a_1^2 + 4k}), a_3\} \geqslant \min\{a_1 + 1, a_3\} \geqslant \frac{1}{2}\varepsilon k$. This implies $u_1(\theta_1) = \theta_1/k \geqslant \frac{1}{2}\varepsilon$. As $2k > (2-\varepsilon)k \geqslant k_2 = kb_1/c_2$, we find $2c_2 > b_1$. Then, by [2, Lemma 5.2], we obtain $D \leqslant 4$. For $D = 3$, it is easy to check $v \leqslant 7k$. If $D = 4$, then by [6, Theorem 5.4.1], $c_3 \geqslant \frac{3}{2}c_2$ and this implies $k_3 = k_2b_2/c_3 \leqslant k_2b_1/(3c_2/2) < \frac{8}{3}k$ and $k_4 < \frac{32}{9}k$, as $b_1/c_2 < 2$. So, $v \leqslant 10k$. Then, by [6, Theorem 4.1.4], the multiplicity m_1 of θ_1 is smaller than $v/(u_1(\theta_1)^2k) < 40/\varepsilon^2$. So, $k < \frac{1}{2}(40/\varepsilon^2 - 1)(40/\varepsilon^2 + 2)$ (by [6, Theorem 5.3.2]). Thus, if we take $\kappa(\varepsilon) = \frac{1}{2}(40/\varepsilon^2 - 1)(40/\varepsilon^2 + 2)$, then $D = 3$ and Γ is either bipartite or a Taylor graph. □

Remark 18. The Hadamard graphs have intersection array $\{k, k-1, \frac{1}{2}k, 1; 1, \frac{1}{2}k, k-1, k\}$ and have $k_2 = 2(k-1)$. For $k = 2^t$ $(t = 1, 2, \ldots)$, there exists a Hadamard graph (see, for example, [6, Section 1.8]). This shows that the above theorem is quite sharp.

Question 19. For fixed positive constant C, are there only finitely many primitive distance-regular graphs with diameter at least three, valency $k \geqslant 3$ and $k_2 < Ck$?

In the next theorem, we classify the distance-regular graphs with $k_2 \leqslant \frac{3}{2}k$ and diameter at least three.

Theorem 20. *Let Γ be a distance-regular graph with diameter $D \geqslant 3$ and valency k. If $k_2 \leqslant \frac{3}{2}k$, then one of the following holds*:

(1) $k = 2$ *and* Γ *is a polygon,*
(2) $D = 3$ *and* Γ *is bipartite,*
(3) $D = 3$ *and* Γ *is a Taylor graph,*
(4) Γ *is the Johnson graph* $J(7,3)$*,*
(5) Γ *is the* 4*-cube.*

Proof. If $c_2 = 1$, then $b_1 = 1$, as $k \leqslant k_2 = kb_1/c_2 \leqslant \frac{3}{2}k$ and this implies $a_1 = k - 2$. As $c_2 = 1$, it follows that $a_1 + 1 = k - 1$ divides k, which gives $k = 2$ and Γ is a polygon. So, we may assume $\frac{1}{2}k \geqslant c_2 \geqslant 2$, as $c_2 > \frac{1}{2}k$ implies that $D = 3$ and Γ is either bipartite or a Taylor graph (by Proposition 5). As the distance-regular line graphs with $c_2 \geqslant 2$ have $D = 2$ (see [6, Theorem 4.2.16]), $c_2 \geqslant 2$ implies that if $a_1 \geqslant \frac{1}{2}k - 1$, then the graph Γ is either a Taylor graph, the Johnson graph $J(7,3)$ or the halved 7-cube by Theorem 16, but the halved 7-cube has $k = 12$ and $k_2 = 35$.

Hence, we may assume $\frac{1}{2}k \geqslant c_2 \geqslant 2$ and $a_1 < \frac{1}{2}k - 1$. This implies $b_1 > \frac{1}{2}k$, and hence $c_2 \geqslant \frac{2}{3}b_1 > \frac{1}{3}k$, as $k_2 \leqslant \frac{3}{2}k$. If $a_1 = 0$, then $\frac{1}{2}k \geqslant c_2 \geqslant \frac{2}{3}(k-1)$, which implies $k \leqslant 4$ and it is easy to check that the theorem holds in this case by [3,7]. So, we may assume $a_1 > 0$ and this implies $k \geqslant 5$, as $a_1 < \frac{1}{2}k - 1$. As $a_1 > 0$, we find $a_2 \geqslant \min\{b_2, c_2\}$, by [6, Proposition 5.5.6], which in turn implies $b_2 < \frac{1}{3}k < c_2$, and hence $D = 3$.

So, from now on, we assume that the diameter D is three, a_1 is positive and the valency k is at least five. Here note that $v \leqslant \frac{7}{2}k$, as $k_3 \leqslant k_2b_2/c_2 < \frac{3}{2}k(k/6)/(k/3) = \frac{3}{4}k$ when $b_2 \leqslant \frac{1}{6}k$, and $b_2 > \frac{1}{6}k$ implies $c_3 > \frac{1}{2}k$ by [6, Theorem 5.4.1], and hence $k_3 \leqslant \frac{3}{2}k(k/3)/(k/2) = k$. As $k_2 \geqslant 2b_1$ and $k_2 \geqslant 2c_3$ (Proposition 5), we find $a_1 \geqslant \frac{1}{4}k - 1$ and $a_3 \geqslant \frac{1}{4}k$ respectively. By [12, Lemma 6], we find $\theta_1 \geqslant \min\{\frac{1}{2}(a_1 + \sqrt{a_1^2 + 4k}), a_3\} \geqslant \min\{a_1 + 1, a_3\} \geqslant \frac{1}{4}k$.

If $m_1 \geqslant \frac{1}{2}k$, then $\frac{7}{2}k^2 \geqslant vk = k^2 + m_1{\theta_1}^2 + m_2{\theta_2}^2 + m_3{\theta_3}^2 \geqslant k^2 + \frac{1}{2}k(\frac{1}{4}k)^2$ implies $k \leqslant 80$. We checked by computer the feasible intersection arrays of distance-regular graphs with diameter three satisfying $k \leqslant 80$ and $k_2 \leqslant \frac{3}{2}k$, and no intersection arrays were feasible.

If $m_1 < \frac{1}{2}k$, then by Lemma 10, $b_1/(\theta_1 + 1) \in \{1, 2\}$, as $\theta_1 \geqslant \frac{1}{4}k$ and $b_1 \leqslant \frac{3}{4}k$. If $b_1/(\theta_1 + 1) = 1$, then there is no such a distance-regular graph by [6, Theorem 4.4.11].

For $b_1/(\theta_1 + 1) = 2$, we first show that one of $m_1 < 48$ and $k \leqslant 510$ holds. If $m_1 \geqslant 48$ and $k > 510$, then by [6, Theorem 4.1.4], $48 \leqslant m_1 = v/(\sum_{i=0}^{3} u_i(\theta_1)^2 k_i) < (7k/2)/((\theta_1/k)^2 k)$, as $u_1(\theta_1) = \theta_1/k$ and $v \leqslant \frac{7}{2}k$. This implies $(\theta_1/k)^2 < \frac{7}{96}$ and hence $\theta_1 < (0.271)k$. As $\frac{1}{2}b_1 - 1 = \theta_1 < (0.271)k$, we have $b_1 < (0.542)k + 2$, which implies $a_1 > (0.458)k - 3$, and hence $a_1 + 1 > (0.458)k - 3 > (0.271)k > \theta_1$, as $k > 510$. This in turn implies $\theta_1 \geqslant a_3$, as $\theta_1 \geqslant \min\{a_1 + 1, a_3\}$. Note that if $k_3 < \frac{1}{2}k$, then $v \leqslant 3k$, as $k_2 \leqslant \frac{3}{2}k$. As $\theta_1 \geqslant \frac{1}{4}k$, we find $m_1 = v/(\sum_{i=0}^{3} u_i(\theta_1)^2 k_i) < 3k/(k/16) = 48$, and this contradicts $m_1 \geqslant 48$. So, $k_3 \geqslant \frac{1}{2}k$ and this implies $b_2 \geqslant \frac{1}{3}c_3$, as $k_3 = k_2b_2/c_3$ and $k_2 \leqslant \frac{3}{2}k$. Then, as $a_3 \leqslant \theta_1 < (0.271)k$, we find $c_3 > (0.729)k$, and hence $b_2 > (0.243)k$. Since $a_2 = k - b_2 - c_2$ and $c_2 > \frac{1}{3}k > (0.333)k$, we find $a_2 < (0.424)k$ and this implies $(0.924)k - 1 > a_1 + a_2 \geqslant k + \theta_2 + \theta_3 \geqslant k - 3 + \theta_3$, as $\theta_1 \geqslant a_3$, and by Lemma 11, we have $\theta_2 \geqslant -3$. So, we obtain $-(0.076)k + 2 \geqslant \theta_3$. As $k > 510$, we have $-(0.07)k > -(0.076)k + 2 \geqslant \theta_3$. Here note $m_1 + m_3 \geqslant k$ by [6, Theorem 4.4.4]. Now $\frac{7}{2}k \geqslant vk \geqslant k^2 + m_1{\theta_1}^2 + m_3{\theta_3}^2$, $\theta_1 \geqslant \frac{1}{4}k$ and $m_1 + m_3 \geqslant k$ imply $k \leqslant 510$. This is a contradiction. So, we find that either $m_1 < 48$ or $k \leqslant 510$ holds. If $m_1 < 48$, then [6, Theorem 5.3.2] implies $k \leqslant 1127$. In conclusion, we find $k \leqslant 1127$.

We checked by computer the feasible intersection arrays of the distance-regular graphs with diameter three satisfying $k_2 \leqslant \frac{3}{2}k$, $\theta_1 = \frac{1}{2}b_1 - 1$, $m_1 < \frac{1}{2}k$ and $k \leqslant 1127$, and no intersection arrays were feasible. □

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